NASA Computational Case Study

The Orbit of Comet ISON

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Computational Algorithms: Modeling, time and coordinate conversion

Abstract

In this case study, we describe a method for computing the position of Sun-grazing Comet ISON for any specific date and time, so that you can find when it will be visible at your location.

1 Introduction

Comet ISON (officially designated C/2012 S1; see Figure 1) was discovered in September 2012 by astronomers using a telescope of the International Scientific Optical Network (ISON) in Russia. It began life at about the time of the formation of the Solar System, some 4.5 billion years ago. Since that time it has been one of a great many icy objects populating the *Oort cloud*—a spherical cloud of bodies surrounding our Solar System, well beyond the orbit of Pluto. [1] Due to random gravitational tugs from other bodies, ISON has now been set on a path on its first-ever—and probably last—trip into the inner Solar System. It will move well inside the orbit of Mercury, and will get to about 1.7 solar radii from the Sun's surface before moving back out away from the Sun toward the Oort cloud again. During its close passage to the Sun around November 28, 2013, Comet ISON may remain intact, break into multiple fragments, or be vaporized altogether; its fate is not yet known. If it survives its firey passage near the Sun, ISON should be easily visible from Earth during December 2013.

With a little effort, you can calculate Comet ISON's position for yourself, and determine when and where it will be visible from your location.

Figure 1: Image of Comet ISON, C/2012 S1, taken on April 10, 2013 by the Hubble Space Telescope. (Credit: NASA, ESA, J.-Y. Li, and the Hubble Comet ISON Imaging Science Team.)

2 Time Measurement

In order to calculate the position of a celestial body like Comet ISON, we begin with finding a suitable method for measuring time. In everyday civil use, we measure time using a rather cumbersome system: years, months (of irregular length, between 28 and 31 days), and day of month. Within a single day, we divide time into hours, minutes, and seconds.

This system is not particularly convenient for use in calculations or plotting, where we would like to have time varying smoothly along some continuous time scale and measured with a single unit. Astronomers have developed such a uniform scale of time measurement, called the *Julian day*: it is defined to be the total number of days elapsed since *noon* on Monday, January 1, 4713 B.C. [2, 3] For example, noon on January 1, 2000 is Julian day 2451545.0, since that's the number of days that had elapsed in the 6712 years since the beginning of 4713 B.C.

The time of day is accounted for by adding a fractional day: for example, 0.0 day for midnight, 0.25 day for 6:00 am, or 0.5 day for noon. Usually the fractional day is counted from midnight *Coordinated Universal Time* (UTC), which is the time in Greenwich, England, and is five hours ahead of Eastern Standard Time (EST).[1] For example, if it is 4:00 am EST on December 25,

[1] Other time scales may be used as well. When it's important to be clear which time scale is used in computing the fractional day, the time scale is added in parentheses following JD: for example, "JD(UTC)".

then add 5 hours to find that this is equivalent to 09:00 UTC December 25. Since $9/24 = 0.375$, 09:00 is 0.375 of a day; hence 09:00 UTC December 25 is the same as December 25.375.

The Julian day for any date on the Gregorian calendar may be found by consulting a table of Julian days [4], or computed using a simple algorithm. Let Y be the year, M the month number (1 for January, 2 for February, etc., up to 12 for December), and D be the day of month (including a fractional day, counted from midnight UTC). The algorithm for computing the Julian day (JD) is [3]:

- If $M = 1$ or $M = 2$, then replace Y with $Y - 1$, and M with $M + 12$. (Otherwise leave M and Y unchanged.)
- Calculate

$$A = \mathrm{INT}\left(\frac{Y}{100}\right) \tag{1}$$

$$B = 2 - A + \mathrm{INT}\left(\frac{A}{4}\right) \tag{2}$$

- Then the Julian day JD is found from

$$\mathrm{JD} = \mathrm{INT}[365.25(Y+4716)] + \mathrm{INT}[30.6001(M+1)] + D + B - 1524.5 \tag{3}$$

Here $\mathrm{INT}(x)$ indicates the greatest integer less than or equal to x.

EXAMPLE 1. Robert Goddard launched his first experimental rocket in Auburn, Massachusetts, on March 16, 1926, at 19:30 UTC. What was the corresponding Julian day?

Solution. The time 19:30 corresponds to a fractional day

$$\frac{19}{24} + \frac{30}{1440} = 0.8125 \text{ day},$$

where we have used 1 day = 24 hours = 1440 minutes. Now using the above algorithm to compute the Julian day, we find:

$$Y = 1926, \quad M = 3, \quad D = 16.8125, \quad A = 19, \quad B = -13,$$

and so

$$\mathrm{JD} = 2425990 + 122 + 16.8125 - 13 - 1524.5 = \boxed{2424591.3125 \quad \mathrm{JD}}$$

Activity 1. *Apollo 11* astronaut Neil Armstrong first set foot on the Moon on July 21, 1969, at 02:56 UTC. Find the Julian day corresponding to this time. (You may solve this problem directly with a calculator, or you may wish to write a general calculator or computer program to convert a calendar date to the corresponding Julian day.)

3 Reference Frames

In order to describe an orbit mathematically, it is necessary to introduce a *reference frame*, with respect to which the orientation of the orbit can be measured. Such a reference frame is determined by a *reference plane*, as well as a *reference direction* within that plane. Two reference planes are in common use:

- The *equator* is the plane of the Earth's equator. This is the reference plane usually chosen for bodies orbiting the Earth.
- The *ecliptic* is the plane of the Earth's orbit. This is the reference plane chosen in most other cases: finding positions of Sun-orbiting bodies such as planets, comets, asteroids, etc.

In both cases, the reference *direction* is chosen to be the direction of the *vernal equinox*, which is in the direction from the Earth to the Sun on the first day of spring in the northern hemisphere. It is a fixed direction in space in the constellation Pisces, and lies along the line of intersection of the planes of the equator and the ecliptic. The line pointing toward the vernal equinox is therefore common to both reference planes.

When an orbiting body crosses the reference plane from *south* to *north*, it is said to be at the *ascending node* of the orbit. Crossing the plane in the other direction, north to south, is the *descending node*.

Throughout this case study, we'll use the ecliptic as the reference plane.

4 Orbital Elements

Comet ISON is in a nearly parabolic orbit around the Sun, and requires some different techniques than those described in an earlier paper for elliptical orbits around the Earth. [5] To calculate the position of the comet at any time, we need several parameters to describe the orbit:

- The size and shape of the orbit are given by the *perihelion distance* q of the parabola—that is, the distance between the comet and the Sun at closest approach. Since the Sun is at the focus of the parabolic orbit, this is the same as the distance from the focus to the vertex of the parabola.
- The *orientation* of the orbit in space with respect to the equatorial reference frame is determined by three angles, as shown in Figure 2:

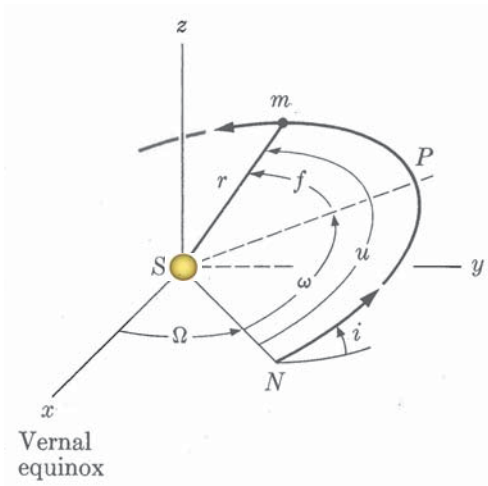

Figure 2: Ecliptic orbital elements of Comet ISON (m) in its parabolic orbit around the Sun (S). Here P is the perihelion point; N is the ascending node of the orbit; Ω is the longitude of the ascending node; ω is the argument of perihelion; and i is the inclination of the orbit. Also shown are the Sun-comet radial distance r; the true anomaly f; and the argument of latitude u. (After McCuskey, 1963 [6].)

  - The *inclination* i of the orbit. This is the dihedral angle between the orbit plane and the plane of the Earth's equator.
  - The *longitude of the ascending node* Ω. This is the angle, measured in the plane of the ecliptic, between the vernal equinox and the ascending node of the orbit.
  - The *argument of perigee* ω. This is the angle, measured in the plane of the orbit, between the ascending node and the perihelion point.

- The four elements given so far completely describe the orbit in space. We now only need to specify *where in the orbit* the comet can be found at a specific time. This is done by specifying the *perihelion time* T_p, which is the time at which the comet is at the perihelion point.

5 Position of ISON Relative to the Sun

We can now begin the orbit calculations, for which we'll use standard two-body orbit propagation methods [6]. Our goal will be to find the azimuth A and elevation h of the comet, as seen from a specific location on the Earth at a given instant of time.

First, we will need to find the *true anomaly*[2] f. This is the angle, measured in the plane of the orbit, from the perihelion point to the comet's position at time t, with the focus of the orbit at the vertex of the angle (Figure 2). The true anomaly f at any time t is found by solving *Barker's equation*,[3]

$$\tan\left(\frac{f}{2}\right) + \frac{1}{3}\tan^3\left(\frac{f}{2}\right) = \sqrt{\frac{GM_\odot}{2q^3}}\,(t - T_p). \tag{4}$$

Here $GM_\odot$ is the product[4] of Newton's gravitational constant G and the mass of the Sun $M_\odot$, and is equal to $1.32712438 \times 10^{20}$ m^3 s^{-2}; q is the perihelion distance, T_p is the perihelion time, and t is the time at which we wish to compute the position of the comet. Both T_p and t are expressed as Julian days. Barker's equation is a cubic equation that can be solved directly, using the method described in Section 8.

Having found f, we now need to compute the radial distance r from the Sun to the comet. This is found from the true anomaly f by writing the plane polar form of the equation for a parabola,

$$r = \frac{2q}{1 + \cos f},$$

and applying the trigonometric identity $\cos^2\theta \equiv (1 + \cos 2\theta)/2$ to get [6]

$$r = q\sec^2\left(\frac{f}{2}\right), \tag{5}$$

where r will have the same units as the given perihelion distance q.

The quantities (r, f) are the polar coordinates of the comet, in the plane of the orbit. But to be able to locate the comet in the sky, we'll need to know the position of the comet relative to the reference plane (the ecliptic). To that end, we will perform a coordinate transformation to go from plane polar coordinates in the plane of the orbit to spherical polar coordinates with the xy plane at the ecliptic. We begin these coordinate transformations by defining the *argument of latitude* u by [6]

$$u = \omega + f, \tag{6}$$

where ω is the given argument of perigee and f is the true anomaly. Then the cartesian coordinates of the comet in the ecliptic frame at time t are given by

$$x = r(\cos u \cos\Omega - \sin u \sin\Omega \cos i) \tag{7}$$

$$y = r(\cos u \sin\Omega + \sin u \cos\Omega \cos i) \tag{8}$$

$$z = r \sin u \sin i \tag{9}$$

[2]The term *anomaly* is used in celestial mechanics to refer to an *angle*. Its use goes back to a time when any departure from uniform circular motion was considered to be "anomalous."

[3]Attributed to Thomas Barker (1722–1809), a meteorologist and astronomer who provided tables of the motion of bodies in parabolic paths in his 1757 monograph *An Account of Discoveries Concerning Comets, with the Way to Find Their Orbits, and Some Improvements in Constructing and Calculating Their Places*. [7]

[4]The *product* $GM_\odot$ is known to better accuracy than either G or $M_\odot$ individually, so for best results, you should use this product in Barker's equation, rather than using individual values for the two constants and multiplying them together.

6 Position of the Sun

We've just found the comet's position at time t relative to the *Sun*, but we want to know where it will be relative to the *Earth*. This requires a second calculation to find the position of the Earth in its orbit around the Sun—or equivalently, the position of the Sun as seen from the Earth. Luckily this can be done with respectable accuracy (to about 1 minute of arc, or $\frac{1}{60}^\circ$) using a set of empirical formulæ[5] [4]. We begin by finding the number of days n elapsed from noon on January 1, 2000 (Julian day 2451545.0) until the time t at which we wish to calculate the comet's position. This can be done by simply finding the difference in Julian days:

$$n = t - 2451545.0 \tag{10}$$

We now compute the *mean longitude* L and *mean anomaly* g of the Sun, using empirical formulæ available in the *Astronomical Almanac* [4]

$$L = 280.460^\circ + 0.9856474\, n \tag{11}$$

$$g = 357.528^\circ + 0.9856003\, n \tag{12}$$

Here L and g will typically be very large angles—over 5000°. You will want to reduce both angles by subtracting 360° repeatedly until they fall within the range of 0° to 360°. (Or, even easier: divide the angle by 360°, and call the result γ. The integer part of γ tells how many complete revolutions there are, so subtract 360° times the integer part of γ from the original angle to reduce it to the proper range.)

Now knowing L and g, we can use another empirical formula to find the *ecliptic longitude* of the Sun, $\lambda_\odot$ [4]:

$$\lambda_\odot = L + 1.915^\circ \sin g + 0.020^\circ \sin 2g \tag{13}$$

and the Earth-Sun distance $R_\odot$:

$$R_\odot = 1.00014 - 0.01671 \cos g - 0.00014 \cos 2g \tag{14}$$

where $R_\odot$ will be in *astronomical units* (AU). One AU is roughly the average distance between the Earth and the Sun, and is equal to $1.49597870 \times 10^{11}$ meters. From $\lambda_\odot$ and $R_\odot$ we can find the cartesian coordinates of the Sun in the ecliptic frame:

$$x_\odot = R_\odot \cos \lambda_\odot \tag{15}$$

$$y_\odot = R_\odot \sin \lambda_\odot \tag{16}$$

$$z_\odot = 0 \tag{17}$$

[5] Small angles are often measured by dividing 1° into 60 *minutes of arc* or *arcminutes*, indicated by the symbol $'$; each arcminute is divided into 60 *seconds of arc* or *arcseconds*, indicated by the symbol $''$. Hence $1^\circ = 60'$ and $1' = 60''$. This system may also be used to express the fractional part of larger angles: for example, $45.5^\circ = 45^\circ 30'$, and $65.375^\circ = 65^\circ 22' 30''$.

7 Position of ISON Relative to the Earth

Having found the position of Comet ISON relative to the Sun (x, y, z) and the Sun relative to the Earth $(x_\odot, y_\odot, z_\odot)$, we can now find the position of Comet ISON relative to the Earth, (x_e, y_e, z_e):

$$x_e = x + x_\odot \tag{18}$$

$$y_e = y + y_\odot \tag{19}$$

$$z_e = z + z_\odot \tag{20}$$

Converting from cartesian to spherical polar coordinates gives an azimuthal angle λ called the *ecliptic longitude*, and a co-polar angle β called the *ecliptic latitude* of the comet: [3]

$$\tan\lambda = \frac{y_e}{x_e} \tag{21}$$

$$\sin\beta = \frac{z_e}{\sqrt{x_e^2 + y_e^2 + z_e^2}} \tag{22}$$

Rotating this result from the plane of the ecliptic to the plane of the equator gives an azimuthal angle α called the *right ascension*, and a co-polar angle δ called the *declination* (Figure 3): [3]

$$\tan\alpha = \frac{\sin\lambda\cos\epsilon - \tan\beta\sin\epsilon}{\cos\lambda} \tag{23}$$

$$\sin\delta = \sin\beta\cos\epsilon + \cos\beta\sin\epsilon\sin\lambda \tag{24}$$

The right ascension α and declination δ are all that are necessary to locate the comet on a star map, since star maps are laid out in these coordinates.

To find the azimuth and elevation of the comet, a few more steps are necessary. We first need to find the *local hour angle* H, which is the angle from the local meridian[6] to the comet, and is found from [3]

$$H = \mathrm{GST} - \Lambda - \alpha \tag{25}$$

Here Λ is the observer's longitude on the Earth (taking *west* longitude as *positive*), and GST is the *Greenwich Sidereal Time*. GST is the angle from the vernal equinox to the prime meridian (measured in the plane of the equator) at time t. The GST (in degrees) may be found from an empirical formula [3]:

$$\begin{aligned}\mathrm{GST} &= 280.46061837 + 360.98564736629(t - 2451545.0) \\ &\quad + 0.000387933\,T^2 - T^3/38710000,\end{aligned} \tag{26}$$

where t is a Julian day, and T is the time in Julian centuries (of 36525 days) from noon, January 1, 2000 (which is Julian day 2451545.0):

$$T = \frac{t - 2451545.0}{36525}. \tag{27}$$

[6] The *local meridian* is the line in the sky running from north to south and passing directly overhead.

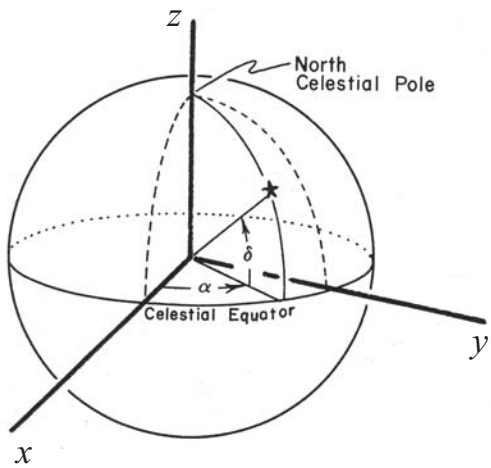

Figure 3: Right ascension and declination of an object, shown here as a star. (After Bate, Mueller, and White, 1971 [8].)

Equation (26) may return very large angles; again you should add or subtract multiples of 360° as needed to bring the GST into the range of 0° to 360°.

Finally, we can find the *azimuth angle* A and *elevation angle* h: [3]

$$\tan A = \frac{\sin H}{\cos H \sin\varphi - \tan\delta \cos\varphi} \tag{28}$$

$$\sin h = \sin\varphi \sin\delta + \cos\varphi \cos\delta \cos H \tag{29}$$

Here φ is the observer's latitude (north positive), and the azimuth A is measured along the horizon, starting from south and going clockwise (to the west). Hence $A = 0°$ is south, $A = 90°$ is west, $A = 180°$ is north, and $A = 270°$ is east. The elevation angle h is measured up from the horizon, so a positive value for h means the comet is above the horizon and is visible.

Now you can predict the position (azimuth and elevation) of Comet ISON at your location, and find the times when it will be visible from your location. All you need are the Julian day for the time t at which you wish to calculate the comet's position, and the comet's orbital elements q, i, Ω, ω, and T_p. To estimate angles, hold your hand at arm's length; your index finger is about 1° wide, and your hand (fingers closed) is about 10° wide. The comet will probably only be visible at night, so choose an observation time between your local sunset and sunrise.

The methods described here can be applied to other comets in parabolic or near-parabolic orbits (i.e. those whose eccentricity is near 1). Other methods must be used for comets in elliptical or hyperbolic orbits, since Barker's equation is only valid for parabolic orbits.

POINTER. Inverse Trigonometric Functions.

Exercise care when computing inverse trigonometric functions, to ensure that the resulting angle is in the correct quadrant. In general, there are *two* correct angles, but the inverse trigonometric function of a calculator or computer will return only one of them. The calculator-provided angle will be between $-90°$ and $+90°$ for the $\sin^{-1}$ and $\tan^{-1}$ functions, and between $0°$ and $180°$ for the $\cos^{-1}$ function.

In cases like Eqs. (21), (23), and (28), where we compute the inverse tangent of the ratio y/x, determining the correct quadrant is simple: if the denominator x is *negative*, then add $180°$ to the calculator's or computer's returned answer; otherwise use the answer as returned. Many computer programming languages provide a special function (typically called something like `atan2(`y, x`)`) just for this type of problem, which will automatically return the angle in the correct quadrant.

In cases like Eqs. (22), (24), and (29) the returned angle is a "latitude angle",and will be between $-90°$ and $+90°$; but since that's where it should be, so no adjustment of the angle is necessary.

Remember that you can always add or subtract as many multiples of $360°$ as you like without changing the angle.

Also, be sure you're clear about whether the computational environment you're using assumes angles are in degrees or radians. Nearly all computer programming languages assume angles are in *radians*. Scientific calculators have a mode setting that allows them to work in either degrees or radians. Computing trigonometric functions and their inverses in the wrong angle mode is one of the most common mistakes people make in doing trigonometric calculations—be sure you don't make this mistake yourself.

8 Solving Barker's Equation

Barker's equation, which we saw earlier, gives the true anomaly f at time t for a parabolic orbit:

$$\tan\left(\frac{f}{2}\right) + \frac{1}{3}\tan^3\left(\frac{f}{2}\right) = \sqrt{\frac{GM_\odot}{2q^3}}\,(t - T_p) \tag{4}$$

(Note that both side of the equation are dimensionless.) We are given time t, the perihelion distance q, and perigee time T_p, and wish to solve for the true anomaly f. In the past, astronomers solved this equation by referring to a set of pre-computed tables called *Barker's tables*. But today, computational resources are plentiful, making it easy to solve Barker's equation directly. The simplest method of solution is a direct method, using the following steps [6] (where K is

the right-hand side of Eq. (4)):

$$\cot s = \frac{3}{2}|K| = \frac{3\sqrt{GM}}{(2q)^{3/2}}\,|t - T_p| \tag{30}$$

$$\cot\left(\frac{s}{2}\right) = \sqrt{1 + \cot^2 s} + \cot s \tag{31}$$

$$\cot w = \sqrt[3]{\cot\left(\frac{s}{2}\right)} \tag{32}$$

$$\cot 2w = \frac{\cot^2 w - 1}{2\cot w} \tag{33}$$

$$\tan\left(\frac{f}{2}\right) = (2\cot 2w) \times \mathrm{sgn}(t - T_0) \tag{34}$$

Here $\mathrm{sgn}(x)$ is the *signum function*, and is defined as

$$\mathrm{sgn}(x) = \begin{cases} -1 & (x < 0) \\ 0 & (x = 0) \\ +1 & (x > 0) \end{cases} \tag{35}$$

Note that both t and T_p will be in Julian days, so the difference $t - T_p$ will be in days. You'll need to convert that to SI units (seconds) before you begin calculating a solution to Barker's equation.

Challenge. Now you can compute the position of the Comet ISON in the sky (azimuth and elevation) at *your* latitude φ and longitude Λ, and a time of your choosing. Find the azimuth A and elevation h of Comet ISON at that time, as observed from your location, given the orbital elements of the comet in Table 1 below.

Table 1. Orbital elements of Comet ISON.

Orbital element	Symbol	Value
Perihelion distance	q	0.0124431 AU
Eccentricity	e	1.0000013
Inclination	i	62.39824°
Longitude of ascending node	Ω	295.65272°
Argument of perihelion	ω	345.56521°
Time of perihelion passage	T_p	2456625.28555 JD

Hint 1. Look up the latitude φ and longitude Λ of your location. Count north latitude and west longitude as positive; south latitude and east longitude are negative.

Hint 2. For the time t, choose some time in early to mid-December 2013, maybe about an hour or so before sunrise or after sunset. Convert this time to Universal Time, and then to a Julian day, using the algorithm described in Section 2. Be sure to include the fractional day as part of the calculation, as was done in Example 1.

Hint 3. Begin your calculations by solving Barker's equation (4) as described in the previous section, then work through Equations (5) through (29) to find the azimuth A and elevation h as your final results.

Hint 4. You might wish to write a computer program to carry out the calculations described here. That will make it easy to change the calculation time t so you can locate the comet at different dates and times.

About the Author

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