NASA Computational Case Study

Modeling Planetary Magnetic and Gravitational Fields

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Abstract

In this case study, we model a planet's magnetic and gravitational fields using spherical harmonic functions. As an exercise, we analyze data on the Earth's magnetic field collected by NASA's MAGSAT spacecraft, and use it to derive a simple magnetic field model based on these spherical harmonic functions.

1 Introduction

There are many times when it is useful to create a *mathematical model* of some physical phenomenon: that is, a set of mathematical equations that summarizes the results of many observations. For example, the Earth has a magnetic field similar to the magnetic field of a bar magnet. Suppose we wish to estimate the magnitude and direction of the Earth's magnetic field at some specific location on the Earth's surface. How would we do that? We *could* search past records for measurements made by various people, hoping to find some that are near the point of interest, then try to interpolate between the observation points. This method would be quite cumbersome, though—it would require sifting through thousands of observations made by many different observers at many different times, trying to find some appropriate observations from which to interpolate.

A simpler method is to create a mathematical model that summarizes all the observations by fitting them to a set of mathematical equations. Once the model has been created, computing an estimate of the Earth's magnetic field at a specific location is easy: just insert the latitude, longitude, and altitude of the location of interest into the equations, and out comes the magnetic field vector.

Such a mathematical model also makes it easy to look for trends in the data with time (the drift of the magnetic poles, for example).

A similar method may be used for modeling the Earth's *gravitational* field. By fitting many observations of the magnitude and direction of the gravitational acceleration to a set of mathematical functions, one may create a mathematical model of the Earth's gravitational field that can be useful for applications such as high-precision orbit predictions.

2 Spherical Harmonics

For modeling the magnetic and gravitational fields of the Earth or other planets, it is customary to use special functions called *spherical harmonics*. In a sense these are two-dimensional counterparts of the sine and cosine functions used in Fourier analysis. Given a set of data defined over the surface of a sphere (such as the Earth), one can fit the data to a series of spherical harmonics in much the same way as one can fit data defined on a circle (i.e. periodic data with a period of 2π radians) to a Fourier series.

Spherical harmonic functions are actually complex-valued functions. Instead of using those directly, we'll create a series using, separately, the real and imaginary components of the spherical harmonics, which are the two sets of functions:

$$\cos(m\phi)\, P_l^m(\cos\theta)$$
$$\sin(m\phi)\, P_l^m(\cos\theta)$$

where θ and ϕ are the usual polar and azimuthal angles (respectively) in spherical polar coordinates, l and m are integer indices with $m \leq l$, and $P_l^m(\cos\theta)$ are special functions called *associated Legendre functions of the first kind*. The first few such Legendre functions (through $l = 3$) are shown in Table 1.

Table 1. Associated Legendre functions, $P_l^m(\cos\theta)$.

$P_0^0(\cos\theta) = 1$
$P_1^0(\cos\theta) = \cos\theta$
$P_1^1(\cos\theta) = \sin\theta$
$P_2^0(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$
$P_2^1(\cos\theta) = \sqrt{3}\sin\theta\cos\theta$
$P_2^2(\cos\theta) = \frac{1}{2}\sqrt{3}\sin^2\theta$
$P_3^0(\cos\theta) = \frac{1}{2}(5\cos^3\theta - 3\cos\theta)$
$P_3^1(\cos\theta) = \frac{1}{4}\sqrt{6}\sin\theta(5\cos^2\theta - 1)$
$P_3^2(\cos\theta) = \frac{1}{2}\sqrt{15}\sin^2\theta\cos\theta$
$P_3^3(\cos\theta) = \frac{1}{4}\sqrt{10}\sin^3\theta$

POINTER. Associated Legendre Functions.

One of the tricky parts about working with spherical harmonic analysis is that there are a number of different normalization conventions for the associated Legendre functions in use, each of which gives rise to different leading coefficients for $P_l^m(\cos\theta)$. The functions shown in Table 1 use the so-called *Schmidt normalization* convention, which is the one most commonly used in geomagnetism. [5] When working in other areas, you may encounter other conventions. The MATLAB function `legendre()` calculates associated Legendre functions, and includes an option that allows you to select among several different normalizations.

Note also that the notation P_l^m indicates a function with two integer indices, l and m; m is *not* an exponent.

3 Magnetic Field Models

Now suppose we wish to model the Earth's magnetic field using spherical harmonics. The Earth's magnetic field is a *vector field*, meaning that there is a magnetic field *vector* associated with each point in space. That complicates our analysis a bit, since it would seem to mean that we have to fit separate series to each of the three components of the magnetic field. But there's a simpler method: suppose we are only interested in modeling the magnetic field *outside* the Earth, due to electric currents inside the Earth's core. Then we can assume there are no electric currents present in the region of space at which we are modeling the magnetic field. With these assumptions, it turns out that Maxwell's equations of electromagnetism allow us to take the Earth's magnetic field vector $\mathbf{B}$ to be the gradient of a *magnetic scalar potential* V: [1–3]

$$\mathbf{B} = -\mu_0 \nabla V, \tag{1}$$

where $\mu_0 = 4\pi \times 10^{-7}$ N A^{-2} is a constant called the *permeability of free space*, the magnetic field $\mathbf{B}$ has units of *teslas* (T), and the magnetic scalar potential V has units of *amperes* (A). The symbol ∇ is the *gradient operator*; in spherical polar coordinates, ∇V is

$$\nabla V(r,\theta,\phi) = \frac{\partial V}{\partial r}\,\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\,\hat{\mathbf{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\,\hat{\mathbf{e}}_\phi, \tag{2}$$

and $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\theta$, and $\hat{\mathbf{e}}_\phi$ are unit vectors in the r, θ, and ϕ directions, respectively. Now all we need to do is fit the magnetic scalar potential V to a *single* spherical harmonic series, which looks like this: [4, 5]

$$V(r,\theta,\phi) = \frac{a}{\mu_0}\sum_{l=1}^{N}\left(\frac{a}{r}\right)^{l+1}\sum_{m=0}^{l}(g_l^m\cos m\phi + h_l^m \sin m\phi)P_l^m(\cos\theta) \tag{3}$$

where $a = 6371.2$ km is the mean radius of the Earth, r is the radial distance from the center of the Earth ($r > a$ since we've assumed we're outside the Earth),

and g_l^m and h_l^m are the expansion coefficients that we need to determine. The $l = 1$ terms represent the dipole components of the magnetic field, $l = 2$ the quadrupole moments, $l = 3$ the octupole moments, and so on.

Activity 1. (a) Why are there are no $l = 0$ terms included in the summations in Eq. (3)? (b) Published tables of coefficients of g_l^m and h_l^m do not include any h_l^0 coefficients. Why would that be?

In Eq. (3), the integer N is called the *order* of the spherical harmonic expansion, and determines how many terms there will be in the series. In general, the more terms, the more accurately the series will represent the data. However, at some point the magnitude of the terms is about the same size as the measurement errors in the data, so it makes little sense to carry the series beyond that point. At the time of this writing, this series is typically carried out to $N = 13$ for the full-scale geomagnetic field model. However, for the purposes of this case study, we'll carry out the series to just $N = 3$ to make things simpler. This will still give a reasonably good model of the Earth's magnetic field; it just won't be as accurate as the $N = 13$ model.

4 Determination of Coefficients

To determine the coefficients g_l^m and h_l^m, let's apply the gradient operator (Eq. (2)) to the spherical harmonic series for the magnetic scalar potential V (Eq. (3)). Each of the components of the resulting vector will give one of the components of the Earth's magnetic field vector. In geomagnetism, we customarily define the three components of the geomagnetic field as:

$$X = -B_\theta = \mu_0(\nabla V)_\theta \quad \text{(northward component)} \tag{4}$$

$$Y = +B_\phi = -\mu_0(\nabla V)_\phi \quad \text{(eastward component)} \tag{5}$$

$$Z = -B_r = \mu_0(\nabla V)_r \quad \text{(downward component)} \tag{6}$$

Let's now calculate the partial derivatives of the potential V using Eqs. (2) and (3) and Table 1, calculating the series out to $N = 2$; this will give eight series

POINTER. Units.

When doing scientific calculations of this sort, it's crucial that you pay proper attention to units of measurement. An easy way to avoid problems with units is to make it a rule that all variables in your computer program will always be stored in base SI units (kilograms, meters, seconds, amperes), and derived units based on them. For this project that means using these units for all calculations inside the program:

- Magnetic field $\mathbf{B}$ in teslas (T).
- Magnetic scalar potential V in amperes (A).
- All lengths (r and a) in meters (m).
- Magnetic dipole moment m in A m^2.
- All angles in radians (rad).
- Coefficients g_l^m and h_l^m in teslas.

If all inputs to your calculations are in base SI units, then the results of the calculations will automatically be in base SI units also. Note, however, that when you look up coefficients g_l^m and h_l^m in the literature, they will be given in *nanoteslas* (nT), so you may wish to multiply your calculated coefficients by 10^9 so that your results are also in nanoteslas.

Remember to convert the MAGSAT magnetic field data to base SI units after reading them from the data file.

terms for each component. The result is

$$\begin{aligned}
X &= \mu_0 \frac{1}{r}\frac{\partial V}{\partial \theta} \\
&= -\frac{a^3}{r^3}\sin\theta g_1^0 + \frac{a^3}{r^3}\cos\phi\cos\theta g_1^1 + \frac{a^3}{r^3}\sin\phi\cos\theta h_1^1 \\
&\quad -3\frac{a^4}{r^4}\cos\theta\sin\theta g_2^0 + \frac{a^4}{r^4}\cos\phi\sqrt{3}(2\cos^2\theta - 1)g_2^1 \\
&\quad +\frac{a^4}{r^4}\sin\phi\sqrt{3}(2\cos^2\theta - 1)h_2^1 + \frac{a^4}{r^4}\cos(2\phi)\sqrt{3}\sin\theta\cos\theta g_2^2 \\
&\quad +\frac{a^4}{r^4}\sin(2\phi)\sqrt{3}\sin\theta\cos\theta h_2^2, \qquad (7) \\
Y &= -\mu_0 \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} \\
&= 0g_1^0 + \frac{a^3}{r^3}\sin\phi g_1^1 - \frac{a^3}{r^3}\cos\phi h_1^1 \\
&\quad +0g_2^0 + \frac{a^4}{r^4}\sin\phi\sqrt{3}\cos\theta g_2^1 \\
&\quad -\frac{a^4}{r^4}\cos\phi\sqrt{3}\cos\theta h_2^1 + \frac{a^4}{r^4}\sin(2\phi)\sqrt{3}\sin\theta g_2^2 \\
&\quad -\frac{a^4}{r^4}\cos(2\phi)\sqrt{3}\sin\theta h_2^2, \qquad (8)
\end{aligned}$$

POINTER. Spherical Polar Coordinates.

You will sometimes encounter different conventions for spherical polar coordinates. As used here, θ is the polar angle (measured from the $+z$ axis), and ϕ is the azimuthal angle (measured in the x-y plane counterclockwise from the $+x$ axis). The (north) latitude is the complement of the polar angle, or $90° - \theta$, while the (east) longitude is the same angle as ϕ.

The coordinate r is the radial distance from the center of the Earth. It is related to the altitude h above sea level by (approximately) $r = a + h$.

$$
\begin{aligned}
Z &= \mu_0 \frac{\partial V}{\partial r} \\
&= -2\frac{a^3}{r^3}\cos\theta g_1^0 - 2\frac{a^3}{r^3}\cos\phi\sin\theta g_1^1 - 2\frac{a^3}{r^3}\sin\phi\sin\theta h_1^1 \\
&\quad -\frac{3}{2}\frac{a^4}{r^4}(3\cos^2\theta - 1)g_2^0 - 3\frac{a^4}{r^4}\cos\phi\sqrt{3}\sin\theta\cos\theta g_2^1 \\
&\quad -3\frac{a^4}{r^4}\sin\phi\sqrt{3}\sin\theta\cos\theta h_2^1 - \frac{3\sqrt{3}}{2}\frac{a^4}{r^4}\cos(2\phi)\sin^2\theta g_2^2 \\
&\quad -\frac{3\sqrt{3}}{2}\frac{a^4}{r^4}\sin(2\phi)\sin^2\theta h_2^2. \qquad (9)
\end{aligned}
$$

Activity 2. Continue calculating derivatives to expand each of these series to order $N = 3$. This will mean seven additional terms (g_3^0, g_3^1, h_3^1, g_3^2, h_3^2, g_3^3, and h_3^3) for a total of 15 terms for each of the X, Y, and Z components. (You may wish to check your results using a symbolic mathematics program such as Mathematica.)

As you can see, the number of terms increases rapidly as the order N is increased. For a spherical harmonic expansion of order N, each of the X, Y, and Z series will have $(N+1)^2 - 1$ terms. For the full $N = 13$ model, that's 195 terms in each series. That's why were limiting the expansion to $N = 3$, for which there are just 15 terms in each series.

Observations of the Earth's magnetic field will take the form of a set of components (X, Y, and Z), measured at a specific polar angle θ (the complement of the latitude), longitude ϕ, and distance from the center of the Earth r. Our data set will then be a set of these variables, and we wish to solve for the unknown g_l^m and h_l^m coefficients.

One fairly straightforward way to do this is to write Eqs. (7) – (9) in matrix form. Doing this to order $N = 2$, we have Eq. (10):

$$
\begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \\ X_2 \\ Y_2 \\ Z_2 \\ X_3 \\ Y_3 \\ Z_3 \\ \vdots \\ X_M \\ Y_M \\ Z_M \end{pmatrix}
=
\begin{pmatrix}
-\frac{a^3}{r_1^3}\sin\theta_1 & \frac{a^3}{r_1^3}\cos\phi_1\cos\theta_1 & \cdots & \frac{a^4}{r_1^4}\sin(2\phi_1)\sqrt{3}\sin\theta_1\cos\theta_1 \\
0 & \frac{a^3}{r_1^3}\sin\phi_1 & \cdots & -\frac{a^4}{r_1^4}\cos(2\phi_1)\sqrt{3}\sin\theta_1 \\
-2\frac{a^3}{r_1^3}\cos\theta_1 & -2\frac{a^3}{r_1^3}\cos\phi_1\sin\theta_1 & \cdots & -\frac{3\sqrt{3}}{2}\frac{a^4}{r_1^4}\sin(2\phi_1)\sin^2\theta_1 \\
\vdots & \vdots & \ddots & \vdots \\
-\frac{a^3}{r_M^3}\sin\theta_M & \frac{a^3}{r_M^3}\cos\phi_M\cos\theta_M & \cdots & \frac{a^4}{r_M^4}\sin(2\phi_M)\sqrt{3}\sin\theta_M\cos\theta_M \\
0 & \frac{a^3}{r_M^3}\sin\phi_M & \cdots & -\frac{a^4}{r_M^4}\cos(2\phi_M)\sqrt{3}\sin\theta_M \\
-2\frac{a^3}{r_M^3}\cos\theta_M & -2\frac{a^3}{r_M^3}\cos\phi_M\sin\theta_M & \cdots & -\frac{3\sqrt{3}}{2}\frac{a^4}{r_M^4}\sin(2\phi_M)\sin^2\theta_M
\end{pmatrix}
\begin{pmatrix} g_1^0 \\ g_1^1 \\ h_1^1 \\ g_2^0 \\ g_2^1 \\ h_2^1 \\ g_2^2 \\ h_2^2 \end{pmatrix}
\tag{10}
$$

Here there are M different data points, each of which consists of an observed magnetic field vector (X_i, Y_i, and Z_i), and the location r_i, θ_i, and ϕ_i at which that observation was made. The left-hand side is a $3M \times 1$ column vector, and contains the observed magnetic field components; let's call this vector $\mathbf{b}$. The matrix on the right of the right-hand side is an 8×1 column vector that contains the coefficients we want to solve for; let's call that vector $\mathbf{g}$. The large matrix on the left of the right-hand side is a $3M \times 8$ matrix that contains each of the terms of the spherical harmonic expansions of X, Y, and Z, evaluated at each location; we'll call this matrix $\mathbf{A}$. The first row of $\mathbf{A}$ times vector $\mathbf{g}$ gives the terms of the spherical harmonic expansion for X, evaluated at the location of data point 1; the second row times $\mathbf{g}$ gives the terms for Y evaluated at the location of data point 1; and the third row times $\mathbf{g}$ gives the terms for Z evaluated at data point 1. Rows 4–6 repeat the pattern: they're the terms in the expansion for X, Y, and Z, but evaluated at the location of data point 2; rows 7–9 are evaluated at the location of data point 3, etc. Then we can write Eq. (10) concisely as

$$\mathbf{b} = \mathbf{A}\mathbf{g}. \tag{11}$$

We know all the magnetic field observations (vector $\mathbf{b}$), and all the spherical harmonic terms evaluated at the data locations (matrix $\mathbf{A}$); we just need to solve for the coefficient vector $\mathbf{g}$. *Formally*, we could do this with a matrix inverse, just by multiplying each side on the left by $\mathbf{A}^{-1}$:

$$\mathbf{g} = \mathbf{A}^{-1}\mathbf{b}.$$

But there's a problem here: matrix $\mathbf{A}$ isn't necessarily square. In fact, it will typically have far more rows than columns, and thus constitute an overdetermined system. How can we compute the inverse of a non-square matrix? Technically we can't, but there is a technique available that allows us to solve Eq. (11) for vector $\mathbf{g}$ in a least-squares sense: it returns the $\mathbf{g}$ that best fits the data in matrix $\mathbf{A}$. This technique is called *singular value decomposition*.

5 Singular Value Decomposition

The singular value decomposition (SVD) of a matrix $\mathbf{A}$ expresses $\mathbf{A}$ as a product of three matrices:

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T, \tag{12}$$

where in our case $\mathbf{A}$ is the matrix of spherical harmonic expansion terms (which is known), and all the matrices on the right-hand side are determined by the SVD method. Singular value decomposition exposes the geometric structure of a matrix, which is an important aspect of many matrix calculations. A matrix can be described as a transformation from one vector space (e.g. vector $\mathbf{A}\mathbf{g}$) to another (e.g. vector $\mathbf{b}$); the components of the SVD quantify the resulting change between the underlying geometry of those vector spaces.

Substituting Eq. (12) into Eq. (11), we have

$$\mathbf{b} = (\mathbf{U}\mathbf{\Sigma}\mathbf{V}^T)\mathbf{g}. \tag{13}$$

In Eqs. (12) and (13), matrices $\mathbf{U}$ and $\mathbf{V}$ are square and orthogonal, in the sense that their column vectors are orthonormal (i.e. $\mathbf{U}^T\mathbf{U} = \mathbf{I}$ and $\mathbf{V}^T\mathbf{V} = \mathbf{I}$, and therefore $\mathbf{U}^{-1} = \mathbf{U}^T$ and $\mathbf{V}^{-1} = \mathbf{V}^T$). The matrix $\mathbf{\Sigma}$ is a generally non-square matrix containing the *singular values* of matrix $\mathbf{A}$ (denoted σ_i) along its main diagonal and zeroes elsewhere. Once the singular value decomposition is performed (i.e. matrices $\mathbf{U}$, $\mathbf{\Sigma}$, and $\mathbf{V}$ have been found), then we can solve Eq. (13) for $\mathbf{g}$:

$$\mathbf{g} = \mathbf{V}\mathbf{\Sigma}^+\mathbf{U}^T\mathbf{b}. \tag{14}$$

Here $\mathbf{\Sigma}^+ \equiv (\mathbf{\Sigma}^T\mathbf{\Sigma})^{-1}\mathbf{\Sigma}^T$ is called the *pseudo-inverse* of matrix $\mathbf{\Sigma}$ [6,7]. Because of the structure of $\mathbf{\Sigma}$ (a non-square diagonal matrix), $\mathbf{\Sigma}^+$ can be found by simply transposing $\mathbf{\Sigma}$ and taking the reciprocals of the elements along its main diagonal, so $\mathbf{\Sigma}^+_{ii} = 1/\sigma_i$, $\mathbf{\Sigma}^+_{ij} = 0$ for $i \neq j$, and $\mathbf{\Sigma}^+\mathbf{\Sigma} = \mathbf{I}$. If $\mathbf{\Sigma}$ is non-singular (i.e. none of the σ_i are near zero), you don't need to do any more. However, if $\mathbf{\Sigma}$ is singular or nearly so (i.e. if any of the σ_i are near zero) you will want to replace the small values of σ_i (those below a specified tolerance) along the main diagonal of $\mathbf{\Sigma}$ by setting the corresponding diagonal elements of $\mathbf{\Sigma}^+$ to zero; this prevents small measurement errors in $\mathbf{b}$ from producing large changes in $\mathbf{g}$. The replacement of the small values in $\mathbf{\Sigma}$ by zeros in $\mathbf{\Sigma}^+$ is equivalent to introducing a smallest cutoff eigenvalue σ_i along the main diagonal of $\mathbf{\Sigma}$ in order for the matrix to be invertible. However, the choice of such cutoff eigenvalues is not unique. As a working solution, the optimal choice of such small eigenvalues can be easily implemented by allowing a threshold ratio between the smallest and the largest eigenvalues which is not smaller than 10 times the machine precision. If this threshold is not met, you increase the cutoff eigenvalues by 10 until it meets the criterion. In all cases you should be able to test the stability of the pseudo-inverse by changing this cutoff eigenvalue.

Eq. (14) is then the solution to our problem. Given the magnetic field observations in vector $\mathbf{b}$ and the singular value decomposition of matrix $\mathbf{A}$ (which gives $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{\Sigma}$), we have the desired coefficients $\mathbf{g}$. But how do we perform the singular value decomposition of $\mathbf{A}$? The simplest method is to use published algorithms [10, 11] or the function `svd()` in MATLAB; the details of the internal workings of these implementations are a bit too complex to go into here.

Activity 3. Derive Eq. (14) from Eq. (13).

Activity 4. Equation (14) is most efficiently calculated by doing the multiplications from right to left. Verify this for yourself by calculating the total number of floating-point multiplications required to compute this equation when the right-hand side is evaluated from right to left, and when it is evaluated from left to right. In Equation (14), if M is the total number of data points, and

the model is of order $N = 3$, then matrix $\mathbf{V}$ is of size 15×15, matrix $\mathbf{\Sigma}^+$ is $15 \times 3M$, matrix $\mathbf{U}^T$ is $3M \times 3M$, and vector $\mathbf{b}$ is $3M \times 1$.

Hint: When multiplying an $m \times n$ matrix by an $n \times p$ matrix, the total number of floating-point multiplications required is $m \times n \times p$.

Challenge. Implement Eq. (14) to order $N = 3$ using MATLAB or some other programming language to find the coefficients g_l^m and h_l^m of the Earth's magnetic field. For data, you can use a set of observations made by the MAGSAT spacecraft during the years 1979–80. MAGSAT collected a magnetic field data point every 0.5 second for 6 months, for a total of about 30 million data points. To make the data a bit more manageable, we've selected roughly 1 point out of every 300,000 (about 1 point every 40 hours) for a total of 100 data points, and made this available as file `magsat-data.dat`. The MAGSAT data is in the form of a plain ASCII text file, in the following format:

Column	Description
1	Time of observation (milliseconds of day; ignore)
2	Latitude $90° - \theta$ (north positive; deg)
3	Longitude ϕ (east positive; deg)
4	Radial distance from center of Earth r (km)
5	Magnetic field component X (nT)
6	Magnetic field component Y (nT)
7	Magnetic field component Z (nT)
8	Attitude processing flag (ignore)

Your program will need to read this data, do any needed unit conversions, compute and store the matrix elements, and then solve the least-squares problem to find the coefficients g_l^m and h_l^m.

If you are using MATLAB, you may wish to investigate MATLAB's built-in pseudo-inverse function `pinv()`, which is more efficient than calculating $\mathbf{\Sigma}^+$ yourself. MATLAB also includes a built-in function `svd()` to perform a singular value decomposition.

Once you have your program working, you may compare your results with the first 15 coefficients of the DGRF 1980 magnetic field model, available from NOAA at `http://www.ngdc.noaa.gov/IAGA/vmod/igrf.html`. Your results will not match DGRF 1980 exactly, because you're not using the same set of observations as input—you're only using 100 points of MAGSAT data. Nevertheless, the comparison will be close enough for you to tell whether you've done the calculation correctly.

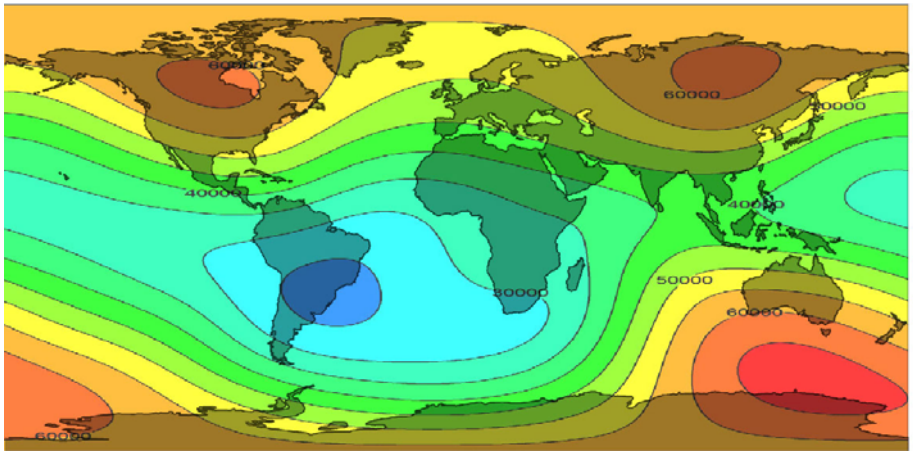

Figure 1: Map of the magnitude of the geomagnetic field at the surface of the Earth at 1980.0, produced from a spherical harmonic model. Contours are labeled in units of nanoteslas (nT). Note that the field is generally strongest (*red*) at the magnetic poles and weakest (*blue*) near the equator. Note also the large dip in magnitude around central South America; this is known as the *South Atlantic Anomaly*.

Challenge. As an additional project, you may wish to try implementing a geomagnetic field model. This would mean writing a computer program to calculate the X, Y, and Z components of the Earth's magnetic field vector at any given latitude, longitude, and radial distance from the center of the Earth, using Eqs. (1) through (3), along with the g_l^m and h_l^m coefficients you just calculated in the previous Challenge. If you have the program calculate the magnetic field vector at many points along a grid over the surface of the Earth, you can produce contour plots of each component. Figure 1 shows something similar: a contour plot of the *magnitude* of the magnetic field vector (i.e. the square root of the sum of the squares of the X, Y, and Z components).

6 Analysis

Besides using them in the magnetic field model, the coefficients g_l^m and h_l^m may also be used to directly calculate a few quantities of interest. For example, the magnetic dipole moment of the Earth's magnetic field can be shown to be given by [12]

$$m = \frac{4\pi a^3}{\mu_0}\sqrt{(g_1^0)^2 + (g_1^1)^2 + (h_1^1)^2}. \tag{15}$$

POINTER. The Inverse Tangent Function.

The inverse tangent function in Eq. (18) may look a bit odd. Why not just cancel the two minus signs? The reason is that the signs of both the numerator and the denominator must be used to ensure that the longitude will be in the correct quadrant.

When computing the inverse tangent of a ratio like this, the rule is: do the division, then compute the arctangent of the quotient. The result, as computed by a calculator or computer, will be between $-90°$ and $+90°$. If the denominator of the original ratio was positive, then use this returned answer; if the denominator was negative, then add $180°$ (π radians) to the returned answer. In Eq. (18) we have to keep the minus signs as they are to ensure that this extra $180°$ is added when the denominator (including the minus sign) is negative.

Most computer programming languages include a special built-in function to handle this case, usually called something like `atan2(`y,x`)`. This will compute the arctangent of y/x, then check the sign of x to place the result in the correct quadrant. Be sure you remember this when computing the arctangent in Eq. (18).

It turns out that the magnetic poles of the Earth are not located at the rotation poles; they are rather located some distance away, and move from one year to the next. We can compute some quantities related to the location of the magnetic poles directly from the coefficients g_l^m and h_l^m. For example, the tilt angle α of the magnetic axis relative to the rotation axis can be shown to be [13]

$$\cos\alpha = \frac{-g_1^0}{\sqrt{(g_1^0)^2 + (g_1^1)^2 + (h_1^1)^2}} \tag{16}$$

The geographic latitude φ_N and longitude λ_N of the magnetic north pole are found from [14]

$$\sin\varphi_N = \frac{-g_1^0}{\sqrt{(g_1^0)^2 + (g_1^1)^2 + (h_1^1)^2}} \tag{17}$$

$$\tan\lambda_N = \frac{-h_1^1}{-g_1^1} \tag{18}$$

Activity 5. Based on your earlier results where you found the magnetic field coefficients g_l^m and h_l^m, determine (a) the magnetic dipole moment of the Earth; (b) the tilt of the magnetic axis; and (c) the latitude and longitude of the magnetic north pole. (All values you calculate will be valid as of 1980, when the MAGSAT observations were made.)

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7 Other Applications

Using the techniques described here, you may also model a planet's *gravitational* field. [15,16] In that case, the measurements are of the acceleration due to gravity $\mathbf{g}$, which can be written as the gradient of a gravitational potential $\mathcal{G}$:

$$\mathbf{g} = -\nabla\mathcal{G} \tag{19}$$

The gravitational potential $\mathcal{G}$ is expanded in a spherical harmonic series, just as as done with the magnetic scalar potential V.

Spherical harmonic expansions have also found applications in plasma physics. It has recently been shown [17] that if you fit a plasma's distribution function to a spherical harmonic series, then the moments of the plasma (i.e. the plasma density, bulk velocity, temperature, and pressure) can be quickly computed as functions of the expansion coefficients. This technique is similar to formulas for calculating the Earth's magnetic dipole moment and magnetic pole location described earlier.

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